THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 2

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different from textbook

Field Axioms of real number:

A1. $a + b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

A2. $a + b = b + a$ if $a, b \in \mathbb{R}$;

A3. $a + (b + c) = (a + b) + c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

A4. There exists $0 \in \mathbb{R}$ such that $a + 0 = a$ for all $a \in \mathbb{R}$;

A5. For any $a \in \mathbb{R}$, there is $b \in \mathbb{R}$ such that $a + b = 0$;

M1. $a \cdot b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

M2. $a \cdot b = b \cdot a$ if $a, b \in \mathbb{R}$;

M3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

M4. There exists $1 \in \mathbb{R} \setminus \{0\}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$;

M5. For any $a \in \mathbb{R} \setminus \{0\}$, there is $b \in \mathbb{R}$ such that $a \cdot b = 1$;

D. $a \cdot (b + c) = a \cdot b + a \cdot c$ if $a, b, c \in \mathbb{R}$.

1. Using the Axioms, show that

   (a) If $a > b$ and $c > 0$, then $c \cdot a > c \cdot b$, and if $a > b$ and $c < 0$, then $c \cdot a < c \cdot b$,

   (b) for all $a \in \mathbb{R} \setminus \{0\}$, $1/(1/a) = a$,

a) Case 1: $c > 0$. So $c \in \mathbb{P}$

$a > b \iff a - b \in \mathbb{P}$

$c \cdot (a-b) \in \mathbb{P}$

Then by D, $c \cdot (a-b)$

$= c \cdot a - c \cdot b \in \mathbb{P}$

So $c \cdot a > c \cdot b$.

Similarly if $c < 0$.

$\frac{1}{\frac{1}{a}}$ is multiplicative inverse of $\frac{1}{a}$.

b) $\frac{1}{\frac{1}{a}} = 1 \cdot \frac{1}{\frac{1}{a}}$ (M4, M2)

$= a \cdot \cancel{\frac{1}{a}} \cdot \cancel{\frac{1}{\frac{1}{a}}}$ (M5)

$= a \cdot 1$ (M5)

$= a$ (M4)

2. Suppose A and B are nonempty subsets of $\mathbb{R}$ that satisfy $a \leq b$ for all $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$.

Pf: let $a \in A$. Then for any $b \in B$, we have

$$a \leq b$$

So b is an u.b. of A, by supremum, we have

$$\sup A \leq b$$

Since b was arbitrary, this means $\sup A$ is a lower bound of B, so by infimum, $\sup A \leq \inf B$.

3. (Exercise 2.4.15 of [BS11]) Show that there exists a positive real number y such that $y^2 = 3$.

Pf: let $S := \{s \in \mathbb{R} : s^2 < 3\}$. "$y = \sup S$"

1st show $\sup S$ exists: · $1 \in S$, so S not empty

· S bounded from above

So by completeness of $\mathbb{R}$, $\sup S$ exists. Moreover, $\sup S \geq 1 > 0$.

Let $y = \sup S$. WTS $y^2 = 3$ by showing $y^2 < 3$, $y^2 > 3$ are not possible.

Sps on the contrary that $y^2 < 3$: Want to find $n \in \mathbb{N}$ s.t. $y + \frac{1}{n} \in S$ (contradicts fact that y is an u.b. of S).

$$\left(y+\frac{1}{n}\right)^2 = y^2 + \frac{2y}{n} + \frac{1}{n^2} \leq y^2 + \frac{2y}{n} + \frac{1}{n} = y^2 + \frac{1}{n}(2y+1).$$

Since $3-y^2$, $3 - y^2 > 0$, $y > 0 \Rightarrow 2y+1 > 0$.

So $\frac{3-y^2}{2y+1} > 0$, so by A.P. $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{3-y^2}{2y+1}$

So
$$\left(y+\frac{1}{n}\right)^2 \leq y^2 + \frac{1}{n}(2y+1) < \cancel{y^2} + \frac{3-\cancel{y^2}}{\cancel{2y+1}}\cancel{(2y+1)} = 3. \quad \lightning$$

$y^2 > 3$: WTS $\exists m \in \mathbb{N}$ s.t. $y - \frac{1}{m}$ is an u.b. of S.

again contradicts fact that y is supremum.

$$y^2 - 3 > 0, \quad 2y > 0, \quad \text{so } \frac{y^2-3}{2y} > 0,$$

So by A.P, $\exists\, m \in \mathbb{N}$ s.t. $\frac{1}{m} < \frac{y^2-3}{2y}$.

$$\left(y - \frac{1}{m}\right)^2 = y^2 - \frac{2y}{m} + \cancel{\frac{1}{m^2}} \geq y^2 - \frac{2y}{m}$$

$$> y^2 - \cancel{2y}\left(\frac{y^2-3}{\cancel{2y}}\right) = 3. \ \lightning$$

So $y^2 = 3$.